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Continuum variable entangled state generated by an asymmetric beam splitter

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Abstract. For an asymmetric beam-splitter a new kind of entangled state $|\eta,\theta\rangle$ is introduced, we then derive the integration measure with which such states can make up a complete and orthonormal representation in two-mode Fock space. We then show how to use $|\eta,\theta\rangle$ in finding new squeezing operator and new squeezed state, whose generation can relies on the asymmetric beamsplitter.

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1 Introduction

Recently, quantum entanglement, which originated from Einstein, Podolsky and Rosen (EPR) in a paper arguing the incompleteness of quantum mechanics [1], is of increasingly interest in studies of quantum information and quantum communication. It lies at the core of some new applications in the emerging field of quantum communication science [2–7]. The concept of entanglement has played a key role in understanding some fundamental problems in quantum mechanics and quantum optics. In an quantum entangled state, a measurement performed on one part of the system provides information on the remaining part, this has now been known as a basic feature of quantum mechanics, though it seems weird. Thus an entangled composite system is non-separable. In EPR's pioneer argument, the entanglement was revealed by explicitly writing the wave function of a bipartite with their relative position X_1-X_2 being x_0 and their total momentum P_1+P_2 being $p_0=0$, i.e. $\psi(x_1,x_2)=(1/2\pi)\int_{-\infty}^{\infty}dpe^{ip(x_1-x_2+x_0)}$. Enlightened by EPR, in reference [8] the simultaneous eigenstate $|\eta\rangle$ of commutative operators (X_1-X_2, P_1+P_2) expressed by two-mode creation operators is found,

$$|\eta\rangle=\exp\left[-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^* a_2^\dagger+a_2^\dagger a_1^\dagger\right]|00\rangle_{12}, \quad (1)$$

where $\eta=(\eta_1+i\eta_2)/\sqrt{2}$ is a complex number, $|00\rangle$ is the two-mode vacuum state, $(a_i,a_i^\dagger)$, $i=1,2$, are two-mode Bose annihilation and creation operators in Fock space, related to (X_i,P_i) by $X_i=(a_i+a_i^\dagger)/\sqrt{2}$, $P_i=(a_i-a_i^\dagger)/(\sqrt{2}i)$. The basic ingredient of the $|\eta\rangle$ state about the coordinate-momentum entanglement can be demonstrated through its disentangling process,

$$|\eta=(\eta_1+i\eta_2)/\sqrt{2}\rangle=e^{-i\eta_1\eta_2/2}\int_{-\infty}^{\infty}dx\,|x\rangle_1\otimes|x-\eta_1\rangle_2\,e^{ix\eta_2}, \quad (2)$$

where $|x\rangle_i$ is the coordinate eigenstate of X_i,

$$|x\rangle_i=\pi^{-1/4}\exp\left[-\frac{1}{2}x^2+\sqrt{2}xa_i^\dagger-\frac{1}{2}a_i^{\dagger 2}\right]|0\rangle_i. \quad (3)$$

Equation (2) shows that once particle 1 is measured in the state $|x\rangle_1$, particle 2 immediately collapses to the coordinate eigenstate $|x-\eta_1\rangle_2$. Equation (2) is named Schmidt decomposition according to reference [9]. On the other hand, the Schmidt decomposition of $|\eta\rangle$ in the two-mode momentum basis is

$$|\eta\rangle=e^{i\eta_1\eta_2/2}\int_{-\infty}^{\infty}dp\,|p\rangle_1\otimes|\eta_2-p\rangle_2\,e^{-i\eta_1 p}, \quad (4)$$

where $|p\rangle_i$ is the momentum eigenvector of P_i,

$$|p\rangle_i=\pi^{-1/4}\exp\left[-\frac{1}{2}p^2+i\sqrt{2}pa_i^\dagger+\frac{1}{2}a_i^{\dagger 2}\right]|0\rangle_i, \quad (5)$$

which tells us that once particle 1 is measured in the state $|p\rangle_1$, particle 2 immediately collapses to the momentum

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eigenstate $|\eta_2 - p\rangle_2$ no matter how far the distance between the two particles is. Thus (2) and (4) together implies the quantum entanglement. Note that the $|\eta\rangle$ states obey the eigenvector equations

$$\left(a_1 - a_2^\dagger\right)|\eta\rangle = \eta\,|\eta\rangle\,, \quad \left(a_2 - a_1^\dagger\right)|\eta\rangle = -\eta^*\,|\eta\rangle\,. \qquad (6)$$

It then follows

$$(X_1 - X_2)\,|\eta\rangle = \eta_1\,|\eta\rangle\,, \qquad (7)$$

$$(P_1 + P_2)\,|\eta\rangle = \eta_2\,|\eta\rangle\,. \qquad (8)$$

The experimental implementation of entangled state of continuous variables does not use the position and momentum of particles but uses light beams that can be characterized by parameters obeying the same commutation relations as position operator X_i and momentum operator P_i. The analogy is based on the fact that a single mode of the quantized radiation field can be expressed in terms of annihilation operators a_i and creation operator $a_i^\dagger$ of a quantum harmonic oscillator with frequency ω, i.e. the electric field operator can be described as $E_i \sim X_i\cos\omega t + P_i\sin\omega t$. It is now known that the EPR light fields with bipartite entanglement can be built from two-single-mode squeezed vacuum state combined at a 50/50 beam splitter [10], i.e. two light fields maximally squeezed in X_i and P_i (in opposite quadratures), respectively entering the two input ports of a 50/50 beamsplitter produce at the output of the beamsplitter a pair of entangled light beams. It is also known that even one single-mode squeezed state incident on a beam splitter yields a bipartite entangled state, because the quantized vacuum field also enters in another input port of the beam splitter and contributes to the two output modes [11].

An interesting and practical question thus naturally arises: if the beamsplitter is not a 50/50 one, but an asymmetric one, then what is the output state when two light fields maximally squeezed in X_i and P_i, respectively entering its two input ports and get superimposed? For an asymmetric beamsplitter without absorption within itself, its complex amplitude reflectivity r and transmissivity t for light incident from one side (or r', t' for light coming from the other side) are not equal to each other. The incident fields (a_1 and a_2), the reflected field a_3 and the transmitted field a_4 may be related by a "scattering matrix" [11]

$$\begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad (9)$$

where t, r, t', and r' obey the reciprocity relations

$$|r'| = |r|\,, \quad |t'| = |t|\,, \quad |r|^2 + |t|^2 = 1,$$
$$r^*t' + r't^* = 0, \quad r^*t + r't'^* = 0, \qquad (10)$$

or the role of a beam splitter operation on two input modes is equivalent to the unitary operator $B \equiv \exp[\theta(a_1^\dagger a_2 - a_2^\dagger a_1)]$, $\theta \neq 0$, (we do not consider the phase difference between the reflected and transmitted fields), with the amplitude reflection and transmission coefficients $t = \cos\theta$, $r = \sin\theta$. The role of B is $Ba_1B^{-1} = a_3$, $Ba_2B^{-1} = a_4$. The details of relationship between two input modes and two output modes for the beam splitter is discussed in [11]. In this work we want to derive the output state for the asymmetric beamsplitter, which turns out to be a new entangled state characteristic of θ. Then we study its main properties and present its application. Our work is arranged as follows: in Sections 2 and 3 we construct the new two-mode entangled state, denoted as $|\eta,\theta\rangle$, which can be generated by an asymmetric beamsplitter. In Section 4 we discuss the orthonormal and completeness relation of $|\eta,\theta\rangle$ and calculate the weight factor for the completeness. In Sections 5 and 6 we show how to apply $|\eta,\theta\rangle$ to deriving new squeezing operator and generalized squeezed state, whose generation can relies on the asymmetric beamsplitter.

2 The new entangled state $|\eta,\theta\rangle$

In the case when two light fields maximally squeezed in X_i and P_i, respectively entering a beam-splitter's two input ports and get superimposed, we find that the output state emerging from asymmetric beam-splitter is

$$|\eta,\theta\rangle = \exp\left\{-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^*\left(a_2^\dagger\sin 2\theta + a_1^\dagger\cos 2\theta\right)\right.$$
$$\left. +\frac{1}{2}\eta^{*2}\cos 2\theta + a_1^\dagger a_2^\dagger\sin 2\theta + \frac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right)\cos 2\theta\right\}|00\rangle\,. \qquad (11)$$

Clearly, when $\theta = \pi/4$, which corresponds to a 50/50 beam-splitter, $|\eta,\pi/4\rangle$ reduces to $|\eta\rangle$. However, it must be clarified that $|\eta,\theta\rangle$ is not a rotated state of $|\eta\rangle$, i.e.,

$$|\eta,\theta\rangle \neq \exp\left[\theta\left(a_1^\dagger a_2 \pm a_2^\dagger a_1\right)\right]|\eta\rangle\,. \qquad (12)$$

Operating a_i, $i = 1, 2$, on $|\eta,\theta\rangle$ respectively gives

$$(a_1 - a_2^\dagger\sin 2\theta - a_1^\dagger\cos 2\theta)|\eta,\theta\rangle = (\eta - \eta^*\cos 2\theta)|\eta,\theta\rangle, \quad (13)$$

and

$$(a_2 - a_1^\dagger\sin 2\theta + a_2^\dagger\cos 2\theta)|\eta,\theta\rangle = -\eta^*\sin 2\theta|\eta,\theta\rangle. \quad (14)$$

From equations (13, 14) we can deduce

$$(a_1\sin 2\theta - a_2\cos 2\theta - a_2^\dagger)|\eta,\theta\rangle = \eta\sin 2\theta|\eta,\theta\rangle, \qquad (15)$$

and

$$(a_1\cos 2\theta + a_2\sin 2\theta - a_1^\dagger)|\eta,\theta\rangle = (\eta\cos 2\theta - \eta^*)|\eta,\theta\rangle. \quad (16)$$

Subtracting (16) from (13) yields

$$(X_2 - X_1\tan\theta)|\eta,\theta\rangle = -\eta_1\tan\theta|\eta,\theta\rangle, \qquad (17)$$

adding (14) and (15) leads to

$$(P_1 + P_2\tan\theta)|\eta,\theta\rangle = \eta_2|\eta,\theta\rangle, \qquad (18)$$

so $|\eta,\theta\rangle$ is the common eigenvector of $(X_2 - X_1\tan\theta)$ and $(P_1 + P_2\tan\theta)$. When $\theta = \pi/4$, equations (17, 18) reduce to equations (7, 8). Therefore, $|\eta,\theta\rangle$ is a new entangled state with a non-trivial expression (see Eq. (11)) and one can Schmidt-decompose it too.

3 The physical meaning of $|\eta,\theta\rangle$ and its relation to an asymmetric beamsplitter

We now explain why the state $|\eta,\theta\rangle$ can describe the production of new entangled light fields using two maximally squeezed light fields in opposite directions (respectively represented by $|p=0\rangle_1$ and $|x=0\rangle_2$) and a non-50/50 beamsplitter, Let the asymmetric beam splitter operator be $\exp\left[2\theta(a_2^\dagger a_1 - a_1^\dagger a_2)\right] \equiv \exp[-2i\theta J_y]$, from

$$\begin{aligned}\exp\left[-2i\theta J_y\right] a_1^\dagger \exp\left[2i\theta J_y\right] &= a_1^\dagger\cos\theta + a_2^\dagger\sin\theta,\\ \exp\left[-2i\theta J_y\right] a_2^\dagger \exp\left[2i\theta J_y\right] &= a_2^\dagger\cos\theta - a_1^\dagger\sin\theta, \qquad (19)\end{aligned}$$

and (3) and (5) we have

$$\begin{aligned}&\exp\left[2\theta(a_2^\dagger a_1 - a_1^\dagger a_2)\right]|p=0\rangle_1\otimes|x=0\rangle_2\\ &=\pi^{-1/2}\exp\left[-2i\theta J_y\right]\exp\left[\frac{1}{2}a_1^{\dagger 2}-\frac{1}{2}a_2^{\dagger 2}\right]\\ &\quad\times\exp\left[2i\theta J_y\right]\exp\left[-2i\theta J_y\right]|00\rangle\\ &=\pi^{-1/2}\exp\left[\frac{1}{2}\left(a_1^\dagger\cos\theta+a_2^\dagger\sin\theta\right)^2\right.\\ &\quad\left.-\frac{1}{2}\left(a_2^\dagger\cos\theta-a_1^\dagger\sin\theta\right)^2\right]\exp\left[2\theta\left(a_2^\dagger a_1-a_1^\dagger a_2\right)\right]|00\rangle\\ &=\exp\left[a_1^\dagger a_2^\dagger\sin 2\theta+\frac{1}{2}\left(a_1^{\dagger 2}-a_2^{\dagger 2}\right)\cos 2\theta\right]\\ &\quad\times|00\rangle=|\eta=0,\theta\rangle\,.\end{aligned}$$

Then operating the displacement operator $D_1(\eta)\equiv\exp[\eta a_1^\dagger-\eta^* a_1]$ on (20) leads to (11), i.e.

$$\begin{aligned}&D_1(\eta)\exp\left[a_2^\dagger a_1^\dagger\sin 2\theta+\frac{1}{2}\left(a_1^{\dagger 2}-a_2^{\dagger 2}\right)\cos 2\theta\right]|00\rangle=\\ &\quad\exp\left\{-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^*\left(a_2^\dagger\sin 2\theta+a_1^\dagger\cos 2\theta\right)\right.\\ &\quad\left.+\frac{1}{2}\eta^{*2}\cos 2\theta+a_1^\dagger a_2^\dagger\sin 2\theta+\frac{1}{2}\left(a_1^{\dagger 2}-a_2^{\dagger 2}\right)\cos 2\theta\right\}|00\rangle\\ &\quad=|\eta,\theta\rangle\,. \qquad (20)\end{aligned}$$

Experimentally, this displacement can be implemented by reflecting the light field of $|\eta=0,\theta\rangle$ from a partially reflecting mirror (say 99% reflection and 1% transmission) and adding through the mirror a field that has been phase and amplitude modulated according to the value $\eta\equiv|\eta|e^{i\Phi}$.

4 Deriving the integration measure with which $|\eta,\theta\rangle$ can make up a complete set

We now examine the main properties of $|\eta,\theta\rangle$. Firstly we see whether the set of $|\eta,\theta\rangle$ is complete. Although it is well known that for any given (pure-state) covariance matrix (CM) the set of Gaussian states with this CM and all possible displacements form an (over)complete set. We still think that this general fact needs to be discussed further in special cases. Because we need to know what is the integration measure with which $|\eta,\theta\rangle$ can make up a complete set. This is like the fact that for the over-completeness relation of coherent state (CS) $|z\rangle$ one needs to demonstrated how to perform the integration $\int d^2z/\pi$ over $|z\rangle\langle z|$, though the CS is defined by displacement transformation and the completeness relation of Fock state $\sum_{n=0}|n\rangle\langle n|=1$ is known. This is also like the fact that although matrices multiplication rule is known, mathematicians and mathematical physicists still want to study miscellaneous matrices which possess special properties. Moreover, in the quantum state engineering of quantum optics physicists have kept trying to discover different kinds of quantum states, the well-known completeness of Fock states $\sum_{n=0}|n\rangle\langle n|=1$ never impede such kind of exploration, though every physical meaningful state can be expanded in terms of $\sum_{n=0}|n\rangle\langle n|=1$. Using the mathematical formula

$$\int\frac{d^2z}{\pi}\exp\left\{\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}\right\}=\frac{1}{\sqrt{\zeta^2-4fg}}\exp\left[\frac{-\zeta\xi\eta+\xi^2 g+\eta^2 f}{\zeta^2-4fg}\right], \qquad (21)$$

$$\begin{aligned}&\mathrm{Re}(\zeta+f+g)<0, \qquad \mathrm{Re}\left(\frac{\zeta^2-4fg}{\zeta+f+g}\right)<0,\\ \text{or}\;\;&\mathrm{Re}(\zeta-f-g)<0, \qquad \mathrm{Re}\left(\frac{\zeta^2-4fg}{\zeta-f-g}\right)<0,\end{aligned}$$

where ζ, f, g are so selected as to insure the integration convergent, and using the normal ordered form of the vacuum projector (: : denotes normal ordering),

$$|00\rangle\langle 00|=:\exp\left\{-a_1^\dagger a_1-a_2^\dagger a_2\right\}:, \qquad (22)$$

as well as the technique of integration within an ordered product (IWOP) of operators [12,13] we can prove that

$$\sin 2\theta \int \frac{d^2\eta}{\pi} |\eta,\theta\rangle\langle\eta,\theta| = \sin 2\theta \int \frac{d^2\eta}{\pi} : \exp\Big\{ -|\eta|^2 + \eta\left(a_1^\dagger - a_2\sin 2\theta - a_1\cos 2\theta\right)$$
$$+\eta^*\left(a_1 - a_2^\dagger \sin 2\theta - a_1^\dagger\cos 2\theta\right) + \frac{1}{2}\left(\eta^2+\eta^{*2}\right)\cos 2\theta$$
$$+\left(a_1^\dagger a_2^\dagger + a_1 a_2\right)\sin 2\theta + \frac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2} + a_1^2 - a_2^2\right)\cos 2\theta - a_1^\dagger a_1 - a_2^\dagger a_2\Big\}:$$
$$=: \exp\Big\{\frac{1}{\sin^2 2\theta}\Big[\left(a_1^\dagger - a_2\sin 2\theta - a_1\cos 2\theta\right)\left(a_1 - a_2^\dagger\sin 2\theta - a_1^\dagger\cos 2\theta\right)$$
$$+\frac{1}{2}\cos 2\theta\left(a_1^\dagger - a_2\sin 2\theta - a_1\cos 2\theta\right)^2 + \frac{1}{2}\cos 2\theta\left(a_1 - a_2^\dagger\sin 2\theta - a_1^\dagger\cos 2\theta\right)^2\Big]$$
$$+\left(a_1^\dagger a_2^\dagger + a_1 a_2\right)\sin 2\theta + \frac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2} + a_1^2 - a_2^2\right)\cos 2\theta - a_1^\dagger a_1 - a_2^\dagger a_2\Big\}:$$
$$=: e^0 := 1. \tag{23}$$

$|\eta,\theta\rangle$ expressed by (11) make up a complete set, i.e.,

see see equation (23) above.

Here the factor $\sin 2\theta$ is needed for the completeness relation, which provides us with a hint that for different special states their integration measures may be different. From (15) and the Hermite conjugate of (14) we have

$$\langle\eta',\theta|\left(a_1\sin 2\theta - a_2\cos 2\theta - a_2^\dagger\right)|\eta,\theta\rangle =$$
$$\eta\sin 2\theta\,\langle\eta',\theta\,|\eta,\theta\rangle = \eta'\sin 2\theta\,\langle\eta',\theta\,|\eta,\theta\rangle. \tag{24}$$

It then follows

$$\sin 2\theta\,(\eta-\eta')\,\langle\eta',\theta\,|\eta,\theta\rangle = 0. \tag{25}$$

Similarly, from (16) and the Hermite conjugate of (13) we derive

$$\langle\eta',\theta|\left(a_1\cos 2\theta + a_2\sin 2\theta - a_1^\dagger\right)|\eta,\theta\rangle =$$
$$(\eta\cos 2\theta - \eta^*)\,\langle\eta',\theta|\,\eta,\theta\rangle = (\eta'\cos 2\theta - \eta'^*)\,\langle\eta',\theta|\,\eta,\theta\rangle,$$
$$[\cos 2\theta\,(\eta-\eta') + (\eta'^* - \eta^*)]\,\langle\eta',\theta\,|\eta,\theta\rangle = 0. \tag{26}$$

Combining the results of equations (25, 26) we obtain

$$\tan 2\theta\,(\eta'^* - \eta^*)\,\langle\eta',\theta\,|\eta,\theta\rangle = 0. \tag{27}$$

As a consequence of (25) and (28) and in reference to (24) we conclude

$$\langle\eta',\theta\,|\eta,\theta\rangle = 2\pi\delta\,(\eta_1-\eta_1')\,\delta\,(\eta_2-\eta_2')\,/\sin 2\theta,$$
$$\eta = (\eta_1 + i\eta_2)/\sqrt{2}. \tag{28}$$

According to Dirac's theory on representation in quantum mechanics, the set of $|\eta,\theta\rangle$ make up a new orthonormal and complete representation in the two-mode Fock space, which is an another entangled state representation. For a review of various applications of the EPR entangled state representation of continuum variables we refer to [14].

5 New squeezing operator derive in terms of $|\eta,\theta\rangle$ and the corresponding squeezed state generated by asymmetric beamsplitter

As an application of the $|\eta,\theta\rangle$ representation, now we construct the following ket-bra operator in an integration form

$$U = \sin 2\theta \int \frac{d^2\eta}{\mu\pi} |\eta/\mu,\theta\rangle\langle\eta,\theta|. \tag{29}$$

where $\eta \to \eta/\mu$ is a c-number dilation transformation. The meaning of discussing (30) lies in generating new squeezed state by an asymmetric beamsplitter. We shall point out that U is a new 2-mode squeezing operator (for a review of squeezed states we refer to [15]). Letting $\mu = e^\lambda$, and using (23) as well as the IWOP technique to perform this integration, we find the normal ordering of U is

see equation (30) below

where we have set $S = \cosh^2\lambda - \cos^2 2\theta$, and

$$M = \frac{\sin 2\theta}{S}\begin{pmatrix} \cosh\lambda\sin 2\theta & \sinh\lambda\cos 2\theta \\ -\sinh\lambda\cos 2\theta & \cosh\lambda\sin 2\theta\end{pmatrix}. \tag{31}$$

$$U = \sin 2\theta \int \frac{d^2\eta}{\mu\pi} : \exp\Big\{ -\frac{1}{2}|\eta|^2\left(1+\frac{1}{\mu^2}\right) + \eta\left(a_1^\dagger/\mu - a_2\sin 2\theta - a_1\cos 2\theta\right) + \eta^*\left(a_1 - a_2^\dagger\sin 2\theta/\mu - a_1^\dagger\cos 2\theta/\mu\right)$$
$$+\frac{1}{2}\left(\frac{1}{\mu^2}\eta^{*2} + \eta^2\right)\cos 2\theta + \left(a_1^\dagger a_2^\dagger + a_1 a_2\right)\sin 2\theta + \frac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2} + a_1^2 - a_2^2\right)\cos 2\theta - a_1^\dagger a_1 - a_2^\dagger a_2\Big\}:$$
$$= \frac{\sin 2\theta}{\sqrt{S}}\exp\left\{\frac{1}{2S}\sinh^2\lambda\cos 2\theta(a_1^{\dagger 2} - a_2^{\dagger 2}) + \frac{1}{2S}a_1^\dagger a_2^\dagger\sinh 2\lambda\sin 2\theta\right\} : \exp\left\{(a_1^\dagger, a_2^\dagger)\,(M-1)\begin{pmatrix}a_1\\a_2\end{pmatrix}\right\}:$$
$$\times\exp\left\{\frac{1}{2S}\sinh^2\lambda\cos 2\theta(a_1^2 - a_2^2) - \frac{1}{2S}a_1 a_2\sinh 2\lambda\sin 2\theta\right\}, \tag{30}$$

Especially, when $\theta = \pi/4$,

$$U_{\theta=\pi/4} = \sec h\lambda \exp\left\{a_1^\dagger a_2^\dagger \tanh\lambda\right\}$$
$$\times : \exp\left\{\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)(\mathrm{sech}\lambda - 1)\right\} : \exp\left\{-a_1 a_2 \tanh\lambda\right\}$$
$$= \int \frac{d^2\eta}{\mu\pi} \left|\eta/\mu\right\rangle \left\langle\eta\right|, \quad (32)$$

where $|\eta/\mu\rangle$ is given by (1), $U_{\theta=\pi/4}$ is the usual two-mode squeezing operator. Equation (33) indicates that the usual two-mode squeezing operator has a neat representation in the entangled state basis [16], this implies that two-mode squeezed state has close relationship with the bipartite entangled state. No wonder the idler mode and the signal mode, which come out of a parametric down-conversion interaction and compose a two-mode squeezed state, are entangled in a frequency domain. The matrix M in (32) can be diagonalized as

$$M = \frac{\sin 2\theta}{S}\begin{pmatrix} 1/2 & i/2 \\ i/2 & 1/2 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad (33)$$

where

$$\alpha = \cosh\lambda \sin 2\theta + i \sinh\lambda \cos 2\theta, \qquad |\alpha| = \sqrt{S},$$
$$\alpha = \sqrt{S} e^{i\varphi}, \qquad \varphi = \tan^{-1}(\tanh\lambda \cot 2\theta), \quad (34)$$

so

$$\ln M = \ln \frac{\sin 2\theta}{S} + \begin{pmatrix} 1/2 & i/2 \\ i/2 & 1/2 \end{pmatrix}\begin{pmatrix} \ln\sqrt{S} + i\varphi & 0 \\ 0 & \ln\sqrt{S} - i\varphi \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \ln\frac{\sin 2\theta}{\sqrt{S}} & \varphi \\ -\varphi & \ln\frac{\sin 2\theta}{\sqrt{S}} \end{pmatrix}. \quad (35)$$

Thus using the operator identity $\exp[a_i^\dagger \Lambda_{ij} a_j] =: \exp\{a_i^\dagger (\Lambda - 1)_{ij} a_j\}:$, where $i, j = 1, 2, \cdots, n$, the repeated indices in a term means summation from 1 to 2, (see Appendix), we have

$$: \exp\left\{\left(a_1^\dagger, a_2^\dagger\right)(M - 1)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right\} :=$$
$$\exp\left\{\left(a_1^\dagger, a_2^\dagger\right)(\ln M)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right\}$$
$$= \exp\left\{\left(a_1^\dagger, a_2^\dagger\right)\begin{pmatrix} \ln\frac{\sin 2\theta}{\sqrt{S}} & \varphi \\ -\varphi & \ln\frac{\sin 2\theta}{\sqrt{S}} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right\}. \quad (36)$$

Using the operator identity

$$\exp[a_i^\dagger \Lambda_{ij} a_j] a_l \exp[-a_i^\dagger \Lambda_{ij} a_j] = \left(e^{-\Lambda}\right)_{lj} a_j, \quad (37)$$

we have

$$U\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}U^{-1} = M^{-1}\left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - K\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}\right], \quad (38)$$

where

$$M^{-1} = \begin{pmatrix} \cosh\lambda & -\cot 2\theta \sinh\lambda \\ \cot 2\theta \sinh\lambda & \cosh\lambda \end{pmatrix},$$
$$K = \frac{\sinh\lambda}{S}\begin{pmatrix} \sinh\lambda\cos 2\theta & \cosh\lambda \sin 2\theta \\ \cosh\lambda \sin 2\theta & -\sinh\lambda \cos 2\theta \end{pmatrix} = \tilde{K}, \quad (39)$$

and

$$M^{-1}K = \begin{pmatrix} 0 & \sinh\lambda/\sin 2\theta \\ \sinh\lambda/\sin 2\theta & 0 \end{pmatrix}, \quad (40)$$
$$M^{-1}\tilde{M}^{-1} = \begin{pmatrix} 1 + (\sinh\lambda/\sin 2\theta)^2 & 0 \\ 0 & 1 + (\sinh\lambda/\sin 2\theta)^2 \end{pmatrix}. \quad (41)$$

One can check the unitarity of U via the following commutative relations,

$$\left[U a_i U^{-1}, U a_j U^{-1}\right] = \left(M^{-1}K\tilde{M}^{-1} - M^{-1}\left(M^{-1}K\right)^T\right)_{ij} = 0,$$
$$\left[U a_i U^{-1}, U a_j^\dagger U^{-1}\right] = \left[M^{-1}\tilde{M}^{-1} - \left(M^{-1}K\right)\left(M^{-1}K\right)^T\right]_{ij} = \delta_{ij}. \quad (42)$$

From (23) and (30) we know that U is a new squeezing operator which squeezes $|\eta, \theta\rangle$ in a natural way,

$$U\left|\eta, \theta\right\rangle = \frac{1}{\mu}\left|\eta/\mu, \theta\right\rangle. \quad (43)$$

6 The property and the generation of the squeezed state generated by U

Writing equation (39) explicitly, we have

$$U a_1 U^{-1} = a_1 \cosh\lambda - a_2 \cot 2\theta \sinh\lambda - a_2^\dagger \csc 2\theta \sinh\lambda,$$
$$U a_2 U^{-1} = a_2 \cosh\lambda + a_1 \cot 2\theta \sinh\lambda - a_1^\dagger \csc 2\theta \sinh\lambda. \quad (44)$$

It then follows

$$U X_1 U^{-1} = \frac{1}{\sqrt{2}} U\left(a_1 + a_1^\dagger\right)U^{-1}$$
$$= X_1 \cosh\lambda - X_2 \cot\theta \sinh\lambda, \quad (45)$$
$$U X_2 U^{-1} = X_2 \cosh\lambda - X_1 \tan\theta \sinh\lambda, \quad (46)$$
$$U P_1 U^{-1} = \frac{1}{\sqrt{2} i} U\left(a_1 - a_1^\dagger\right)U^{-1}$$
$$= P_1 \cosh\lambda + P_2 \tan\theta \sinh\lambda, \quad (47)$$
$$U P_2 U^{-1} = P_2 \cosh\lambda + P_1 \cot\theta \sinh\lambda, \quad (48)$$

so under the U transformation the two quadratures for two-mode optical field become

$$U(X_1 + X_2)U^{-1} = X_1(\cosh\lambda - \tan\theta \sinh\lambda) + X_2(\cosh\lambda - \cot\theta\sinh\lambda), \quad (49)$$
$$U(P_1 + P_2)U^{-1} = P_1(\cosh\lambda + \cot\theta\sinh\lambda) + P_2(\cosh\lambda + \tan\theta\sinh\lambda). \quad (50)$$

$$\exp\left[-2i\theta J_y\right] e^{\frac{1}{2}a_1^{\dagger 2}\tanh\lambda} e^{-\frac{1}{2}a_2^{\dagger 2}\tanh\lambda} \exp\left[2i\theta J_y\right]\exp\left[-2i\theta J_y\right]|00\rangle$$
$$= \exp\left\{\frac{\tanh\lambda}{2}\left[\left(a_1^\dagger\cos\theta + a_2^\dagger\sin\theta\right)^2 - \left(a_2^\dagger\cos\theta - a_1^\dagger\sin\theta\right)^2\right]\right\}|00\rangle$$
$$= \exp\left\{\frac{\tanh\lambda}{2}\cos 2\theta (a_1^{\dagger 2} - a_2^{\dagger 2}) + a_1^\dagger a_2^\dagger \tanh\lambda\sin 2\theta\right\}|00\rangle\,, \quad (58)$$

Using (31) we know that $U^{-1} = U^\dagger$ generates the $\theta-$related squeezed vacuum state,

$$U^{-1}|00\rangle = \frac{\sin 2\theta}{\sqrt{S}}\exp\left\{\frac{\cos 2\theta}{2S}\sinh^2\lambda\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right) - \frac{\sin 2\theta}{2S}a_1^\dagger a_2^\dagger \sinh 2\lambda\right\}|00\rangle \equiv |\,\rangle_{\lambda,\theta}\,. \quad (51)$$

The expectation value of the two quadratures in the state $|\,\rangle_{\lambda,\theta}$ are

$${}_{\lambda,\theta}\langle\,|(X_1+X_2)|\,\rangle_{\lambda,\theta} = 0, \quad {}_{\lambda,\theta}\langle\,|(P_1+P_2)|\,\rangle_{\lambda,\theta} = 0, \quad (52)$$

thus the variance of the two quadratures are

$${}_{\lambda,\theta}\left\langle \Delta(X_1+X_2)^2\right\rangle_{\lambda,\theta} = {}_{\lambda,\theta}\langle\,|(X_1+X_2)^2|\,\rangle_{\lambda,\theta}$$
$$= \langle 00|U(X_1+X_2)^2U^{-1}|00\rangle$$
$$= \cosh^2\lambda + \frac{\sinh^2\lambda}{2}\left(\tan^2\theta + \cot^2\theta\right) - \frac{\sinh 2\lambda}{2}(\tan\theta + \cot\theta)\,, \quad (53)$$
$${}_{\lambda,\theta}\left\langle \Delta(P_1+P_2)^2\right\rangle_{\lambda,\theta} = {}_{\lambda,\theta}\langle\,|(P_1+P_2)^2|\,\rangle_{\lambda,\theta}$$
$$= \langle 00|U(P_1+P_2)^2U^{-1}|00\rangle$$
$$= \cosh^2\lambda + \frac{\sinh^2\lambda}{2}\left(\tan^2\theta + \cot^2\theta\right) + \frac{\sinh 2\lambda}{2}(\tan\theta + \cot\theta)\,. \quad (54)$$

Especially, when $\theta = \pi/4$, this $\theta-$related squeezed vacuum state reduces to the usual two-mode squeezed state, (54) and (55) respectively become

$${}_{\lambda,\pi/4}\left\langle \Delta(X_1+X_2)^2\right\rangle_{\lambda,\pi/4} = e^{-2\lambda},$$
$${}_{\lambda,\pi/4}\langle\,|(P_1+P_2)^2|\,\rangle_{\lambda,\pi/4} = e^{2\lambda}, \quad (55)$$

as expected. On the other hand, due to $\tan^2\theta + \cot^2\theta \geqslant 2$, $\tan\theta + \cot\theta \geqslant 2$, from (55) we see

$${}_{\lambda,\theta}\left\langle \Delta(P_1+P_2)^2\right\rangle_{\lambda,\theta} \geqslant (\cosh\lambda + \sinh\lambda)^2 = e^{2\lambda}, \quad (56)$$

which means that the $\theta-$related squeezed vacuum state can exhibit more stronger squeezing in one quadrature than that of the usual two-mode squeezed vacuum state. Finally, since $\sin 2\theta \le 1$, $\cos^2 2\theta \le 1$, when the squeezing parameter $\mu = e^\lambda$ is large enough such that $\cosh^2\lambda \gg \cos^2 2\theta$, $S = \cosh^2\lambda - \cos^2 2\theta \sim \cosh^2\lambda$, then $U|00\rangle$ is approximately equal to (up to a constant factor)

$$U|00\rangle \to \exp\left\{\frac{\tanh^2\lambda}{2}\cos 2\theta\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right) + a_1^\dagger a_2^\dagger\tanh\lambda\sin 2\theta\right\}|00\rangle\,. \quad (57)$$

Experimentally, this state can be approximately produced when two light fields respectively squeezed in X_i and P_i with the same squeezing parameter $\mu = e^\lambda$, expressed by $e^{\frac{1}{2}a_1^{\dagger 2}\tanh\lambda}|0\rangle_1$ and $e^{-\frac{1}{2}a_2^{\dagger 2}\tanh\lambda}|0\rangle_1$ respectively, entering the asymmetric beamsplitter's two input ports and get superimposed, then using (19) we know that the output state is

see equation (58) above

which is approximately equal to (58) when $\tanh^2\lambda \sim \tanh\lambda$.

In summary, as a non-trivial generalization of the fact that a 50/50 beamsplitter can produce an EPR entangled state, we see that two light fields maximally squeezed in opposite quadratures, respectively entering two input ports of a non-50/50 beamsplitter and get superimposed, will produce at the output a pair of entangled light beams, which can be ideally expressed by the new entangled state $|\eta,\theta\rangle$. Such states are potentially useful, because they make up a complete and orthonormal representation in two-mode Fock space as Dirac's theory stated [17]. Using $|\eta,\theta\rangle$ we have derived new squeezing operator and squeezed state (52) which in some approximation can be generated by an asymmetric beamsplitter. The foundation of $|\eta,\theta\rangle$ generalizes the EPR entangled state representation with continuous variables. For the 3-mode squeezed state which relates to the corresponding entangled state representation we refer to [18].

Appendix

Here we prove the operator identity

$$\exp\left[a_i^\dagger \Lambda_{ij} a_j\right] =: \exp\{a_i^\dagger(\Lambda - 1)_{ij}a_j\}:, \quad (59)$$

which can lead to (37) from (36). The proof is as follows. Using the completeness relation of the $n-$mode coherent state $|\mathbf{z}\rangle = |z_1, \cdots, z_n\rangle$, i.e.

$$\int \prod_i \left[\frac{d^2 z_i}{\pi}\right] |\mathbf{z}\rangle \langle \mathbf{z}| = 1, \tag{60}$$

and (38) as well as $\exp[-a_i^\dagger \Lambda_{ij} a_j] |\mathbf{0}\rangle = |\mathbf{0}\rangle$, by virtue of the IWOP technique, we have

$$\begin{aligned}\exp\left[a_i^\dagger \Lambda_{ij} a_j\right] &= \int \prod_i \left[\frac{d^2 z_i}{\pi}\right] \exp\left[a_i^\dagger \Lambda_{ij} a_j\right] \exp\left(a_i^\dagger z_i\right) \\ &\quad \times \exp\left[-a_i^\dagger \Lambda_{ij} a_j\right] |\mathbf{0}\rangle \langle \mathbf{z}| \exp\left(-\frac{1}{2}|z_i|^2\right) \\ &= \int \prod_i \left[\frac{d^2 z_i}{\pi}\right] \exp\left[a_i^\dagger \left(e^\Lambda\right)_{il} z_l\right] |\mathbf{0}\rangle \langle \mathbf{z}| \exp\left(-\frac{1}{2}|z_i|^2\right) \\ &= \int \prod_i \left[\frac{d^2 z_i}{\pi}\right] : \exp\left[-|z_i|^2 + a_i^\dagger \left(e^\Lambda\right)_{il} z_l + z_i^* a_i - a_i^\dagger a_i\right] : \\ &= : \exp\{a_i^\dagger \left(e^\Lambda - 1\right)_{ij} a_j\} : . \end{aligned} \tag{61}$$

One of the authors, Hong-yi Fan, considers that this work is in memory of Prof. L. Mandel, one of the pioneers of quantum optics, who friendly invited him to visit University of Rochester in 1987 and discussed with him on squeezed states, entangled states and the existence of creation operator's eigenket.

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